Bi-differential calculus and the KdV equation

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Abstract

A *gauged bi-differential calculus* over an associative (and not necessarily commutative) algebra $\mathcal{A}$ is an $\mathbb{N}_0$-graded left $\mathcal{A}$-module with two covariant derivatives acting on it which, as a consequence of certain (e.g., nonlinear differential) equations, are flat and anticommute. As a consequence, there is an iterative construction of generalized conserved currents. We associate a gauged bi-differential calculus with the Korteweg-de-Vries equation and use it to compute conserved densities of this equation.

1 Introduction

A distinguishing feature of soliton equations and other completely integrable models is the existence of an infinite set of conservation laws. For the special case of two-dimensional (principal) chiral or σ-models, a simple iterative construction of conserved currents and charges had been presented in [1]. In [2, 3, 4] some generalizations of this work in the framework of noncommutative geometry have been achieved. In a recent work [5], the existence of an infinite set of conserved currents in several completely integrable classical models, including chiral and Toda models, as well as the KP and self-dual Yang-Mills equations, has been traced back to a simple construction of an infinite chain of closed (respectively, covariantly constant) 1-forms in a (gauged) bi-differential calculus. A bi-differential calculus consists of a graded algebra on which two anticommuting differential maps act. In a gauged bi-differential calculus these maps are extended to covariant derivatives which, as a consequence of, e.g., nonlinear differential equations, are flat and anticommuting.

Section 2 introduces a mathematical scheme which may be regarded as the crucial structure behind the appearance of an infinite chain of conserved currents in the abovementioned completely integrable models (see also [5]). Section 3 shows how to realize such a scheme in terms of bi-differential calculi and covariant derivatives. Section 4 treats the case of the Korteweg-de-Vries equation in some detail. Section 5 contains some conclusions.

2 The central mathematical construction

Let $\mathcal{A}$ be an associative algebra over $\mathbb{R}$ or $\mathbb{C}$ with a unit $\mathbb{1}$. In the following, a *linear* map is meant to be linear over $\mathbb{R}$, respectively $\mathbb{C}$. We consider an $\mathbb{N}_0$-graded left $\mathcal{A}$-module $\mathcal{M} = \sum_{r\geq 0} \mathcal{M}^r$, on which two linear maps $\mathrm{D}, \mathcal{D} : \mathcal{M}^r \to \mathcal{M}^{r+1}$ act such that

$$\mathrm{D}^2 = 0 \,, \quad \mathcal{D}^2 = 0 \,, \quad \mathcal{D}\,\mathrm{D} = g\,\mathrm{D}\,\mathcal{D} \tag{2.1}$$

with some $g \in \mathcal{A}$. Furthermore, we assume that, for some $s > 0$, there is a (nonvanishing) $\chi^{(0)} \in \mathcal{M}^{s-1}$ with

$$\mathcal{D}\chi^{(0)} = 0 \,. \tag{2.2}$$

Then

$$J^{(1)} = \mathrm{D}\chi^{(0)} \tag{2.3}$$

is $\mathcal{D}$-closed, i.e.,

$$\mathcal{D}J^{(1)} = g\,\mathrm{D}\mathcal{D}\chi^{(0)} = 0 \,. \tag{2.4}$$

If every $\mathcal{D}$-closed element of $\mathcal{M}^s$ is $\mathcal{D}$-exact, then

$$J^{(1)} = \mathcal{D}\chi^{(1)} \tag{2.5}$$

with some $\chi^{(1)} \in \mathcal{M}^{s-1}$. Now let $J^{(m)} \in \mathcal{M}^s$ satisfy

$$\mathcal{D}J^{(m)} = 0 \,, \qquad J^{(m)} = \mathrm{D}\chi^{(m-1)} \,. \tag{2.6}$$

Then

$$J^{(m)} = \mathcal{D}\chi^{(m)} \tag{2.7}$$

with some $\chi^{(m)} \in \mathcal{M}^{s-1}$ (which is determined only up to addition of some $\beta \in \mathcal{M}^{s-1}$ with $\mathcal{D}\beta = 0$), and

$$J^{(m+1)} = \mathrm{D}\chi^{(m)} \tag{2.8}$$

is also $\mathcal{D}$-closed:

$$\mathcal{D}J^{(m+1)} = g\,\mathrm{D}\mathcal{D}\chi^{(m)} = g\,\mathrm{D}J^{(m)} = g\,\mathrm{D}^2\chi^{(m-1)} = 0\,. \tag{2.9}$$

In this way one obtains an infinite tower of $\mathcal{D}$-closed elements $J^{(m)} \in \mathcal{M}^s$ and elements $\chi^{(m)} \in \mathcal{M}^{s-1}$ which satisfy

$$\mathcal{D}\chi^{(m+1)} = \mathrm{D}\chi^{(m)}\,. \tag{2.10}$$

In certain cases this construction may break down at some level $m > 0$ or become trivial in some sense (see also [5]). In terms of

$$\chi = \sum_{m=0}^{\infty} \lambda^m \chi^{(m)} \tag{2.11}$$

with a parameter λ, the set of equations (2.10) leads to

$$\mathcal{D}\chi = \lambda\,\mathrm{D}\,\chi\,. \tag{2.12}$$

Conversely, if the last equation holds for all λ, we recover (2.10).

3 Bi-differential calculi and covariant derivatives

In this section we consider realizations of the structure introduced in the last section in terms of covariant exterior derivatives.

Definition 1. A *graded algebra* over $\mathcal{A}$ is an $\mathbb{N}_0$-graded associative algebra $\Omega(\mathcal{A}) = \bigoplus_{r\geq 0} \Omega^r(\mathcal{A})$ such that $\Omega^0(\mathcal{A}) = \mathcal{A}$ and the unit $\mathbb{I}$ of $\mathcal{A}$ extends to a unit of $\Omega(\mathcal{A})$, i.e., $\mathbb{I}\,w = w\,\mathbb{I} = w$ for all $w \in \Omega(\mathcal{A})$.

Definition 2. A *differential calculus* $(\Omega(\mathcal{A}), \mathrm{d})$ over $\mathcal{A}$ consists of a graded algebra $\Omega(\mathcal{A})$ over $\mathcal{A}$ and a linear map $\mathrm{d} : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A})$ with the properties

$$\mathrm{d}^2 = 0\,, \tag{3.13}$$
$$\mathrm{d}(w\,w') = (\mathrm{d}w)\,w' + (-1)^r\,w\,\mathrm{d}w' \tag{3.14}$$

where $w \in \Omega^r(\mathcal{A})$ and $w' \in \Omega(\mathcal{A})$.[1] We also require that d generates $\Omega(\mathcal{A})$ in the sense that $\Omega^{r+1}(\mathcal{A}) = \mathcal{A}\,(\mathrm{d}\Omega^r(\mathcal{A}))\,\mathcal{A}$.

Definition 3. A triple $(\Omega(\mathcal{A}), \mathrm{d}, \delta)$ consisting of a graded algebra $\Omega(\mathcal{A})$ over $\mathcal{A}$ and two linear maps $\mathrm{d}, \delta : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A})$ with the properties (3.13), (3.14) and

$$\delta\,\mathrm{d} + \mathrm{d}\,\delta = 0 \tag{3.15}$$

[1]The identity $\mathbb{I}\mathbb{I} = \mathbb{I}$ then implies $\mathrm{d}\mathbb{I} = 0$.

is called a *bi-differential calculus*.

Let $(\Omega(\mathcal{A}), \mathrm{d}, \delta)$ be a bi-differential calculus, and A, B two $N \times N$-matrices of 1-forms (i.e., the entries are elements of $\Omega^1(\mathcal{A})$). We introduce

$$\mathrm{D} = \mathrm{d} + A \qquad \mathcal{D} = \delta + B \tag{3.16}$$

which act from the left on $N \times M$-matrices with entries in $\Omega(\mathcal{A})$. The latter form an $\mathbb{N}_0$-graded left $\mathcal{A}$-module $\mathcal{M} = \bigoplus_{r \geq 0} \mathcal{M}^r$. Then the conditions (2.1) with $g = -1$ can be expressed in terms of A and B as follows,

$$\mathrm{D}^2 = 0 \qquad \Longleftrightarrow \qquad F = \mathrm{d}A + AA = 0\,, \tag{3.17}$$

$$\mathcal{D}^2 = 0 \qquad \Longleftrightarrow \qquad \mathcal{F} = \delta B + BB = 0\,, \tag{3.18}$$

$$\mathrm{D}\,\mathcal{D} + \mathcal{D}\,\mathrm{D} = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}B + \delta A + BA + AB = 0\,. \tag{3.19}$$

If these conditions are satisfied, we speak of a *gauged bi-differential calculus*.

If $B = 0$, the conditions (3.17)-(3.19) become $F = 0$ and $\delta A = 0$. There are two obvious ways to further reduce the latter equations:

(i) We can solve $F = 0$ by setting $A = g^{-1}\,\mathrm{d}g$ with an invertible $N \times N$-matrix g with entries in $\mathcal{A}$. The remaining equation reads $\delta(g^{-1}\,\mathrm{d}g) = 0$ which resembles the field equation of principal chiral models (cf [5]).

(ii) We can solve $\delta A = 0$ via $A = \delta\phi$ with a matrix ϕ. Then we are left with the equation $\mathrm{d}(\delta\phi) + (\delta\phi)^2 = 0$ which generalizes the so-called 'pseudodual chiral models' (cf [6] and references cited there).

4 Example: conserved densities of the Korteweg-de-Vries equation

Let $\mathcal{A}_0 = C^\infty(\mathbb{R} \times \mathcal{I})$ be the algebra of smooth functions of coordinates t, x, where $\mathcal{I}$ is an interval, and $\mathcal{A}$ the noncommutative algebra generated by the elements of $\mathcal{A}_0$ and the partial derivative $\partial_x = \partial/\partial x$ such that $\partial_x f = f_x + f\,\partial_x$ for $f \in \mathcal{A}$. Here, f_x denotes the partial derivative of f with respect to x. Furthermore, let $\Omega^1(\mathcal{A})$ be the $\mathcal{A}$-bimodule generated by two elements τ and ξ which commute with all elements of $\mathcal{A}$. With

$$\tau\,\xi = -\xi\,\tau\,, \quad \tau\,\tau = 0 = \xi\,\xi \tag{4.1}$$

we obtain a graded algebra $\Omega(\mathcal{A}) = \bigoplus_{r=0}^{2} \Omega^r(\mathcal{A})$ over $\mathcal{A}$. Now

$$\begin{aligned} \mathrm{d}f &= [\partial_t + 4\partial_x^3, f]\,\tau - 6\,[\partial_x^2, f]\,\xi \\ &= (f_t + 4f_{xxx} + 12\,f_{xx}\,\partial_x + 12\,f_x\,\partial_x^2)\,\tau - 6\,(f_{xx} + 2\,f_x\,\partial_x)\,\xi\,, \end{aligned} \tag{4.2}$$

$$\delta f = -\frac{1}{2}\,[\partial_x^2, f]\,\tau + [\partial_x, f]\,\xi = -\frac{1}{2}\,(f_{xx} + 2\,f_x\,\partial_x)\,\tau + f_x\,\xi \tag{4.3}$$

and

$$\mathrm{d}(f\,\tau + h\,\xi) = (\mathrm{d}f)\,\tau + (\mathrm{d}h)\,\xi\,, \quad \delta(f\,\tau + h\,\xi) = (\delta f)\,\tau + (\delta h)\,\xi \tag{4.4}$$

define two linear maps $\mathrm{d}, \delta : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A})$, and $(\Omega(\mathcal{A}), \mathrm{d}, \delta)$ becomes a bi-differential calculus over $\mathcal{A}$.

Remark. The above calculus is *noncommutative* in the sense that differentials do not, in general, commute with elements of $\mathcal{A}$, even with those of the commutative subalgebra $\mathcal{A}_0$. In particular, we have $x\,\delta x = (\delta x)\,x + \tau$. A (noncommutative) differential calculus is a basic structure in 'noncommutative geometry'. ■

With $B = 0$ and $A \in \Omega^1(\mathcal{A})$, (3.19) becomes $\delta A = 0$ which is solved by

$$A = \delta v = -\frac{1}{2}(v_{xx} + 2\,v_x\,\partial_x)\,\tau + v_x\,\xi \tag{4.5}$$

with $v \in \mathcal{A}_0$. Then $F = 0$ takes the form

$$v_{tx} + v_{xxxx} - v_x\,v_{xx} = 0\,. \tag{4.6}$$

With the substitution

$$u = -v_x \tag{4.7}$$

this becomes the *Korteweg-de-Vries* equation

$$u_t + u_{xxx} + u\,u_x = 0\,. \tag{4.8}$$

Let $\mathcal{M} = \Omega(\mathcal{A})$. The general solution of $\delta\chi^{(0)} = 0$ for $\chi^{(0)} \in \mathcal{A}$ is

$$\chi^{(0)} = \sum_{n=0}^{\infty} c_n(t)\,\partial_x^n \tag{4.9}$$

with functions c_n depending on t only. A particular solution is given by $\chi^{(0)} = 1$. The equation (2.12) with $\mathcal{D} = \delta$ is equivalent to the two equations

$$\chi_x = -\lambda\,(6\,\chi_{xx} + u\,\chi + 12\,\chi_x\,\partial_x)\,, \tag{4.10}$$

$$-\frac{1}{2}\,\chi_{xx} = \lambda\left(\chi_t + 4\,\chi_{xxx} + \frac{1}{2}\,u_x\,\chi + u\,\chi_x + 6\,\chi_{xx}\,\partial_x\right)\,. \tag{4.11}$$

With

$$\chi = \sum_{n=0}^{\infty} \chi_n\,\partial_x^n \tag{4.12}$$

the first equation is turned into the following set of equations,

$$\chi_{0,x} + \lambda\,(6\,\chi_{0,xx} + u\,\chi_0) = 0 \tag{4.13}$$

$$\chi_{n,x} + \lambda\,(6\,\chi_{n,xx} + 12\,\chi_{n-1,x} + u\,\chi_n) = 0 \quad (n > 0)\ . \tag{4.14}$$

Inserting[2]

$$\chi_0 = e^{-\lambda\,\varphi}\,, \qquad \varphi = \sum_{m=0}^{\infty} (6\lambda)^m\,\varphi^{(m)} \tag{4.15}$$

(which sets $\chi^{(0)} = 1$) in (4.13), we get

$$\varphi_x = u - 6\,\lambda\,\varphi_{xx} + 6\,\lambda^2\,(\varphi_x)^2 \tag{4.16}$$

which in turn leads to

$$\varphi_x^{(0)} = u\,, \qquad \varphi_x^{(1)} = -u_x \tag{4.17}$$

and

$$\varphi_x^{(m)} = -\varphi_{xx}^{(m-1)} + \frac{1}{6}\sum_{k=0}^{m-2} \varphi_x^{(k)}\,\varphi_x^{(m-2-k)} \tag{4.18}$$

for $m > 1$. Hence

$$\varphi_x^{(2)} = u_{xx} + \frac{1}{6}\,u^2\,, \tag{4.19}$$

$$\varphi_x^{(3)} = -(u_{xx} + \frac{1}{3}\,u^2)_x\,, \tag{4.20}$$

$$\varphi_x^{(4)} = \frac{1}{6}\,[\frac{1}{3}\,u^3 - (u_x)^2] + [u_{xxx} + \frac{1}{2}\,(u^2)_x]_x\,, \tag{4.21}$$

$$\varphi_x^{(5)} = -[\frac{4}{27}\,u^3 + \frac{5}{6}\,(u_x)^2 + \frac{4}{3}\,u\,u_{xx} + u_{xxxx}]_x\,, \tag{4.22}$$

$$\begin{aligned}\varphi_x^{(6)} &= \frac{5}{216}\,[u^4 - 12\,u\,(u_x)^2 + \frac{36}{5}\,(u_{xx})^2] \\ &\quad +[u_{xxxxx} + \frac{5}{3}\,u\,u_{xxx} + \frac{5}{6}\,u^2\,u_x + 3\,u_x\,u_{xx}]_x\,,\end{aligned} \tag{4.23}$$

$$\begin{aligned}\varphi_x^{(7)} &= -[\frac{2}{27}\,u^4 + \frac{4}{3}\,u^2\,u_{xx} + \frac{5}{3}\,u\,(u_x)^2 + \frac{14}{3}\,u_x\,u_{xxx} + 2\,u\,u_{xxxx} \\ &\quad +\frac{10}{3}\,(u_{xx})^2 + u_{xxxxxx}]_x\,,\end{aligned} \tag{4.24}$$

[2] Direct use of the expansion (2.11) for χ_0 leads to *nonlocal* conserved densities. The transformation from χ_0 to φ and subsequent expansion of φ leads to *local* expressions, however.

$$
\begin{aligned}
\varphi_x^{(8)} &= \frac{7}{648}\,[u^5 - 30\,u^2(u_x)^2 + 36\,u\,(u_{xx})^2 - \frac{108}{7}\,(u_{xxx})^2] \\
&\quad + [u_{xxxxxxx} + \frac{7}{3}\,u\,u_{xxxxx} + \frac{20}{3}\,u_x\,u_{xxxx} + \frac{35}{3}\,u_{xx}\,u_{xxx} \\
&\quad + \frac{35}{18}\,u^2\,u_{xxx} + \frac{95}{54}\,(u_x)^3 + \frac{35}{216}\,(u^4)_x + \frac{7}{2}\,(u^2)_x u_{xx}]_x\,, \qquad (4.25) \\
\varphi_x^{(9)} &= -[\frac{16}{405}\,u^5 + \frac{20}{9}\,u^2\,(u_x)^2 + \frac{32}{27}\,u^3\,u_{xx} + \frac{113}{9}\,(u_x)^2\,u_{xx} + \frac{80}{9}\,u\,(u_{xx})^2 \\
&\quad + \frac{112}{9}\,u\,u_x\,u_{xxx} + \frac{8}{3}\,u^2\,u_{xxxx} + \frac{23}{2}\,(u_{xxx})^2 + \frac{56}{3}\,u_{xx}\,u_{xxxx} \\
&\quad + 9\,u_x\,u_{xxxxx} + \frac{8}{3}\,u\,u_{xxxxxx} + u_{xxxxxxxx}]_x\,, \qquad (4.26) \\
\varphi_x^{(10)} &= \frac{7}{1296}\,[u^6 - 60\,u^3\,(u_x)^2 + 108\,u^2\,(u_{xx})^2 - 30\,(u_x)^4 - \frac{648}{7}\,u\,(u_{xxx})^2 \\
&\quad + \frac{720}{7}\,(u_{xx})^3 + \frac{216}{7}\,(u_{xxxx})^2] + [u_{xxxxxxxxx} + \frac{35}{72}\,u^4\,u_x + \frac{35}{18}\,u^3\,u_{xxx} \\
&\quad + \frac{21}{2}\,u^2\,u_x\,u_{xx} + \frac{95}{18}\,u\,(u_x)^3 + \frac{7}{2}\,u^2\,u_{xxxxx} + 20\,u\,u_x\,u_{xxxx} \\
&\quad + \frac{455}{18}\,(u_x)^2\,u_{xxx} + 35\,u\,u_{xx}\,u_{xxx} + \frac{69}{2}\,u_x\,(u_{xx})^2 + 3\,u\,u_{xxxxxxx} \\
&\quad + \frac{35}{3}\,u_x\,u_{xxxxxx} + 28\,u_{xx}\,u_{xxxxx} + \frac{125}{3}\,u_{xxx}\,u_{xxxx}]_x \qquad (4.27)
\end{aligned}
$$

and so forth. As a consequence of (4.10) and (4.11), we have

$$
\chi_{0,t} + \chi_{0,xxx} + \frac{1}{2}\,u\,\chi_{0,x} = 0\,. \qquad (4.28)
$$

In terms of φ this reads

$$
\varphi_t + \varphi_{xxx} - 3\,\lambda\,\varphi_x\,\varphi_{xx} + \lambda^2(\varphi_x)^3 + u\,\varphi_x/2 = 0 \qquad (4.29)
$$

and application of ∂_x leads to a conservation law for φ_x,

$$
\varphi_{xt} = -(\varphi_{xxx} - 3\,\lambda\,\varphi_x\,\varphi_{xx} + \lambda^2(\varphi_x)^3 + u\,\varphi_x/2)_x\,. \qquad (4.30)
$$

Hence, the $\varphi_x^{(m)}$ obtained above are conserved densities of the KdV equation. Let

$$
Q^{(m)} = \int_{\mathcal{I}} \varphi_x^{(m)}\,dx \qquad (4.31)
$$

where dx is the ordinary (Lebesgue) measure on $\mathbb{R}$. Here we assume that either u is periodic in x or that u and its x-derivatives vanish sufficiently rapidly at the (finite or infinite) boundaries

of the interval $\mathcal{I}$, so that the above integrals exist (see also [7]). Note that $\varphi^{(m)}$ will not, in general, be periodic or vanish at the ends of the interval, however. Now we have

$$\frac{d}{dt}Q^{(m)} = \int_{\mathcal{I}} \varphi^{(m)}_{xt} \, dx = 0 \,. \tag{4.32}$$

Neglecting x-derivatives (which do not contribute to (4.31)) in the expressions for $\varphi^{(m)}_x$, we observe that $Q^{(m)} = 0$ for odd m. The nonvanishing conserved charges are

$$\begin{aligned}
Q^{(0)} &= \int_{\mathcal{I}} u \, dx && (4.33)\\
Q^{(2)} &= \frac{1}{6} \int_{\mathcal{I}} u^2 \, dx && (4.34)\\
Q^{(4)} &= \frac{1}{6} \int_{\mathcal{I}} [\frac{1}{3} u^3 - (u_x)^2] \, dx && (4.35)\\
Q^{(6)} &= \frac{5}{216} \int_{\mathcal{I}} [u^4 - 12\, u\, (u_x)^2 + \frac{36}{5} (u_{xx})^2] \, dx && (4.36)\\
Q^{(8)} &= \frac{7}{648} \int_{\mathcal{I}} [u^5 - 30\, u^2\, (u_x)^2 + 36\, u\, (u_{xx})^2 - \frac{108}{7} (u_{xxx})^2] \, dx && (4.37)\\
Q^{(10)} &= \frac{7}{1296} \int_{\mathcal{I}} [u^6 - 60\, u^3\, (u_x)^2 + 108\, u^2\, (u_{xx})^2 - 30\, (u_x)^4 \\
&\quad -\frac{648}{7} u\, (u_{xxx})^2 + \frac{720}{7} (u_{xx})^3 + \frac{216}{7} (u_{xxxx})^2] \, dx && (4.38)
\end{aligned}$$

and so forth. The integrands are in agreement (up to irrelevant constant factors) with $T_1, \ldots, T_6$ in [7], equations (5a)-(10a). Using computer algebra, it is easy to compute higher conserved charges. In [8] the uniqueness of the above sequence of conserved polynomial densities of the Korteweg-de-Vries equation has been shown. Therefore, the remaining freedom in the above construction cannot lead to additional polynomial conserved densities.

Remark. Application of the central construction in section 2 to the case under consideration requires that δ-closed elements of $\Omega^1(\mathcal{A})$ are δ-exact. $J \in \Omega^1(\mathcal{A})$ can be written as $J = a\,\tau + b\,\xi$ with $a, b \in \mathcal{A}$. Then $\delta J = 0$ means $a + b_x/2 + b\,\partial_x = c$, where $c_x = 0$. Introducing $\chi(t,x) = \int^x b(t,x')\,dx'$, we have $J = (c - \frac{1}{2}\,\chi_{xx} - \chi_x\,\partial_x)\,\tau + \chi_x\,\xi = \delta\chi + c\,\tau$. This does not work, however, for periodic boundary conditions on $\mathcal{I}$ (so that $\mathcal{I}$ is actually replaced by the circle S^1), since the indefinite integral of a periodic function b need not be periodic. We still have the problem that the 1-form τ is not δ-exact in $\Omega(\mathcal{A})$. But with an extension of $\mathcal{A}$ and $\Omega(\mathcal{A})$ (see also [5], section 5.3) it becomes exact. This amounts to setting $\tau = \delta y$ with an additional coordinate y. Then δ-closed elements of $\mathcal{M}^1$ are indeed δ-exact. ■

5 Conclusions

The existence of a gauged bi-differential calculus as a (non-trivial) consequence of certain (e.g., differential, difference, or operator) equations may turn out to be a common feature of completely integrable systems. In [5] we have demonstrated that this concept covers many of the known soliton equations and other (in some sense) integrable models. The relation with various notions of complete integrability and approaches towards a classification of integrable models still has to be explored further. Moreover, the notion of a (gauged) bi-differential calculus and its generalization considered in section 2 applies to a large variety of structures (based on noncommutative algebras) most of which are far away from classical completely integrable models. It generalizes a characteristic feature of such models, namely the existence of an infinite set of conserved currents, into a framework of noncommutative geometry where an appropriate notion of complete integrability according to our knowledge is not yet at hand.

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